Doctoral Thesis Review Report

Title: Problems of Numerical Calculation of Derivatives and Differential Equations of Fractional Orders
Institution: Institute of Applied Computer Science, Lodz University of Technology, Faculty of Electrical, Electronic, Computer and Control Engineering, Poland
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Reviewer: J. A. Tenreiro Machado, Coordinator Professor with Habilitation, Dept. of Electrical Engineering, Institute of Engineering, Polytechnic of Porto, Portugal

Review Report

The document is organized in introduction, three main chapters, and summary and conclusions, namely:

- Introduction
- Chapter 1, Fractional Order Derivatives and Integrals
- Chapter 2, Accuracy Problems of Fractional Order Derivatives and Integrals Numerical Calculations
- Chapter 3, Numerical Approximation of the Inverse Laplace Transform
- Summary and Conclusions.

The document includes also 3 large appendices entitled:

- Practical Applications
- Fundamentals of Numerical Integration and Differentiation
- Computing Environment Configuration.

Besides these parts we find also List of Used Symbols and Abbreviations, Abstract, Introduction, Bibliography, List of Figures, and List of Tables.
The Doctoral Thesis is written in English and makes a total of 162 pages.

The research work addresses the area of Fractional calculus (FC). FC represents the generalization of the concept of derivative operator from integer order to arbitrary order, that can be a real or a complex. value. Derivatives and integrals of integer order are special cases of fractional calculus. Since most present day human knowledge is based on the classical integer-order integrodifferential calculus, there is a wide area of possible generalization by the application of the tools of FC. Furthermore, it is today recognized that FC allows the accurate modeling of many natural and artificial phenomena with long range memory effects. Nevertheless, the application of these concepts only emerged in the last years, being one of the reasons the difficulties in calculating fractional derivatives and integral in applied sciences, namely when involving numerical data. FC gained a considerable popularity and is presently a topic with a strong development.

The main body of the work is devoted to several aspects of numerical calculus and computer programming when involving computations within FC.

The Introduction describes the scope of the work, the theses formulated by the author and the contents of the PhD dissertation. In the reviewer opinion the length and organization are adequate. As often occurs in PhD works, we can discuss the inclusion of extra text about the history and development of the area. However, the option of the author seems

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adequate since presently FC is no longer an "exotic" area and the inclusion of extra descriptions would lead to a lengthy and unnecessary long text.

The chapter 1 entitled "Fractional Order Derivatives and Integrals" includes some FC fundamentals and is sub-divided into three sections (The Definitions of Fractional Order Derivatives and Integrals, Exact Values Formulas, Useful Mathematical Functions and Their Computation). These topics are undoubtedly necessary to introduce at the initial part of the work. We can obviously discuss its possible length extension to include another issues or, alternatively, to move it to the end as another appendix. The reviewer believes that the structure and length are adequate, being concise and assertive, so that readers can focus attention on what is going to be discussed.

The chapter 2 entitled "Accuracy Problems of Fractional Order Derivatives and Integrals Numerical Calculations" is sub-divided into four sections (Methods of Accuracy Increase for Application with the Riemann-Liouville /the Caputo Formulas, Methods of Accuracy and Efficiency Increase for the Grünwald-Letnikov Formula, Versatile Accuracy Assessment Criteria, Numerical Accuracy of Non-integer Order ($v > 1$) Derivatives and Integrals Computations). In this case we have a very long chapter that addresses several important distinct topics. We find not only the analysis of different formulations for the fractional operators, but also assessment several numerical methods, the presentation of criteria and the calculation of fractional high orders. All these topics are in fact usually avoided in the existing literature due to many mathematical subtle paradoxes. The reviewer notes for example the on-going and long-standing discussion about the pros and cons of the distinct formulations of the operators and the differences between the results emerging from them, both in analytical, numerical and interpretation points of view. Given the large volume of new results the reviewer considers that the division into 2, or even 3, parts would lead to a more manageable text, but, on the other hand, understand the author's option toward a structure oriented by the theses presented. Therefore, the reviewer believes that the author spent a considerable effort also in the organization of this chapter and that its structure is adequate for readers to follow.

The chapter 3 entitled "Numerical Approximation of the Inverse Laplace Transform" is sub-divided into four sections (Introduction to the LaplaceTransform, Introduction to the Inverse Laplace Transform, Numerical Approximation of the Inverse Laplace Transform, The Accuracy of Numerical Approximation of the Inverse Laplace, Transform for Solutions of Linear Fractional Differential Equations). There are several comments that the reviewer would like to forward. First we can discuss if the introductory parts in the 2 initial sections would be better moved toward either chapter 1 or to an appendix. The reviewer considers that the author's option to be the most appropriate both to avoid having the same topic scattered along distant parts of the work and to avoid the multiplication of appendices. Moreover, this chapter has a manageable length and represents another theses developed by the author. The remaining part of the chapter presents numerical schemes to to the reviewer best knowledge are formulated for the first time in the area of FC. Therefore, in this perspective, we can discuss if this chapter would be the topic of a second PhD thesis. Also, we can extend it by discussing and analyzing the adequacy of Laplace transform versus Fourier transform or even other tools such as Z-transform, windowed Fourier transform or wavelet transform, just to mention the most well-known. As usual with good scientific work the research avenues opened are always larger that those explored. Having these ideas in mind, the reviewer considers that this chapter is an important contribution to those substantively presented previously.

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The Summary and Conclusions explores and discusses adequately the main results of the work. Its extension and writing style are adequate. In the reviewer opinion the author could expand (and be more ambitious) in the part about future research directions. On the other hand, to have parsimony in the possible future works is also a quality that the reviewer would like to remark.

The authors includes three appendices. Appendix A, entitled "Practical Applications" with two parts (Mitigation of Time and Memory Shortages Influence on the Accuracy of Real-time Microprocessor Calculations, The Double Exponential Quadrature As a Replacement for Major Gauss Quadratures), appendix B, entitled "Fundamentals of Numerical Integration and Differentiation" with three parts (Newton-Cotes Quadratures, Gauss Quadratures., Central (Finite) Differences) and appendix C entitled "Computing Environment Configuration" with four parts (Precisions of Floating Point Arithmetic, Python vs C++, Computer System Details, Computational Accuracy Measure). These appendices have considerable and valuable information that somehow is required in chapters 2 and 3. Again, we can discuss about its possible inclusion directly into the chapter itself. We should note that the thesis addresses topics involving integrodifferential calculus, transforms, numerical analysis, and computational issues. Most of these issues are not common and need many sources to be synthesized in a readable form. Therefore, in the reviewer opinion the existence of 3 appendices represent a good compromise between having a large number of appendices and large chapters, that would lead to some lack of focus in the main chapters. In a different viewpoint, the reviewer would like to emphasize that these appendices have very significant contents that would easily lead to relevant chapters in any PhD thesis and, may be, to good journal review papers. In fact, a careful review demonstrates that, besides the main chapters, additional research with high-level results can be found in the appendices.

In summary, the reviewer considers that the author presents a considerable volume of work that, in fact, could be distributed along several different doctoral thesis. The aspects discussed are novel, and discussed assertively and in detail. The symbols and formulas are clearly defined, and the figures and tables are informative. The analysis of the state of the art is adequate and the list of references includes many relevant contributions. Moreover, the text and writing style are fluent, so that readers can follow closely the ideas presented by the author.

The reviewer believes that the doctoral thesis represents a timely and important contribution to the scientific knowledge in the area. In fact, previously published books and journal papers address limited areas which lead to a lack of engineering tools for studying fractional-order systems. The research developed by Dariusz Brzeziński integrates tackles several areas that are often considered separately under the umbrellas of mathematical methods and computer science. Therefore, algorithmic and computational issues are studied alongside with differential calculus, transforms, or modeling and control. The reviewer highlights, for example, the novelties on the programming component of the work, as for example, the solution to many numerical problems by applying modern programming technology called the infinite precision computing. The results of this strategy are visible in the excellent results achieved provide a bright road for future researchers and engineers to follow in FC and applied sciences.

Bearing these ideas in mind, the reviewer considers that the submitted Doctoral Thesis represents an **excellent work** that should be **accepted** in the present form and deserves **distinction**.